Calculus in mathematics for economists

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Conference

The Learning and Teaching of Calculus Across Disciplines

Bergen, 5 - 9 June 2023

• • •

With an reaction by Frank Feudel (HU Berlin & khdm)

Remarks on this post-conference version of the presentation •

- This version of the slides is a slight revision of the slides presented at the conference.
- Mainly the appendices (which I did not use in the presentation) were integrated in the main part of the presentation.
- Added sections, subsections and slides are marked with an •.
- Also attached were Frank Feudel's slides on his reaction.

Rainer Vosskamp

- 1983 - 1989:
  - Studies in mathematics (major)
  - Studies in economics (minor)
- 1989 - 2008:
  - Economics at universities and economic research institutes
  - Teaching: macroeconomics, microeconomics, public economics etc.
  - Research: economics
- Since 2008:
  - $\approx$ senior lecturer and $\approx$ associate professor
  - University of Kassel
  - Teaching:
    - mathematics for economists
    - macroeconomics, evolutionary economics, economics of education etc.
  - Research: empirical educational research (sometimes)

Table of Contents

Section 1

Introduction

Heterogeneity in ME (cf. *Voßkamp* (2023c))

Business Adminstration and economics

- Two disciplines:
  - Business adminstration
  - Economics
- Business adminstration:
  - Focus on a single firm
  - Example: What drives the price setting of a firm?
  - Importance of mathematics: low, increasing
- Economics:
  - Focus on (a part of) a wohle economy
  - Example: What drives the growth of gross domestic product (GDP)?
  - Importance of mathematics: high, increasing
- Here: mathematics for economists (including mathematics for business adminstration)

Definitions

Definition 1 (ME, CE)

1. *ME := mathematics in economics.*
2. *CE := calculus in economics.*

Cornerstones

- Focus: first-year ME
- Mostly: broad picture of CE
- Partly: details on CE taken from my ME lectures

Section 2

The framework of mathematics for economists

2 The framework of mathematics for economists

Subsection 1

Mathematics in Economics

Mathematics in economics

- Important role of mathematics in economics for about 70 years
- Fundamental works (e.g.):
  - *Samuelson* (1947)
  - *Arrow/Debreu* (1954)
- Increasing use of mathematics in journal articles:
  - *Stigler et al.* (1995)
- Handbook of mathematical economics (vol. 1 - 4):
  - *Arrow* (1981, 1982, 1986, 1991)

Calculus in economics

- In particular in mainstream economics ("neoclassical economics")
  - Physics, mechanics
- Background:
  - Equilibria:
    - Static
    - Dynamic
  - Rational agents:
    - Households maximize utility.
    - Firms maximize profit.
- An alternative paradigm: evolutionary economics
  - Biology
- However, calculus play an important role in economics, generally.

ME in economics study programs (cf. *Voßkamp* (2017))

- One to three compulsory courses ME in the first year
- Moreover: one to three compulsory courses in statistics
- Conception ME:
  - Independent and equal module
  - Support module
- CE is typically integrated in ME.

Subsection 2

The students

The students (cf. *Laging/Voßkamp* (2017))

- A rough characterization:
  1. Many first-year students are surprised that mathematics modules are compulsory.
  2. First-year students' knowledge of secondary mathematics is low on average.
  3. Many students choose a study program in business administration due to an extrinsic motivation.
  4. Many students aim to pass the exam in ME, not to achieve good grades.
- Economics students (and in particular business administration students) are different . . . .

Prior knowledge in calculus

- Basic differential and integral calculus is subject of secondary school mathematics.
- Binding educational standards of the states
- Prior knowledge in calculus is (very) low on average.
  - cf. *Laging/Voßkamp* (2017), *Büchele/Feudel* (2023)

Differential calculus at school according to binding educational standards of the KMK •

- Students will be able to
  - interpret the derivative as a local rate of change in particular.
  - describe rates of change functionally (derivative function) and interpret them.
  - derive the functions of secondary level I, also using the factor and sum rule.
  - use the product rule to derive functions.
  - use the derivative to determine monotonicity and extrema of functions.
  - develop the derivative graph from the function graph and vice versa."
- *KMK* (2012)

Integral calculus at school according to binding educational standards of the KMK •

- Students will be able to
  - interpret the definite integral, in particular as a (re)constructed stock.
  - geometrically-visually justify the main theorem as the relationship between derivative and integral notions.
  - to integrate functions by means of root functions.
- *KMK* (2012)

Subsection 3

The teachers

The ME Teachers (cf. *Voßkamp* (2017))

- Many teachers of ME
  - are educated mostly in economics.
  - are affiliated with economics departments.
  - are (non-tenured) lecturers (workload 4 to 18 h / week).
  - are not involved in research projects.
- High degree of heterogeneity among ME teachers (in German-speaking countries)
- ME teachers are different . . .

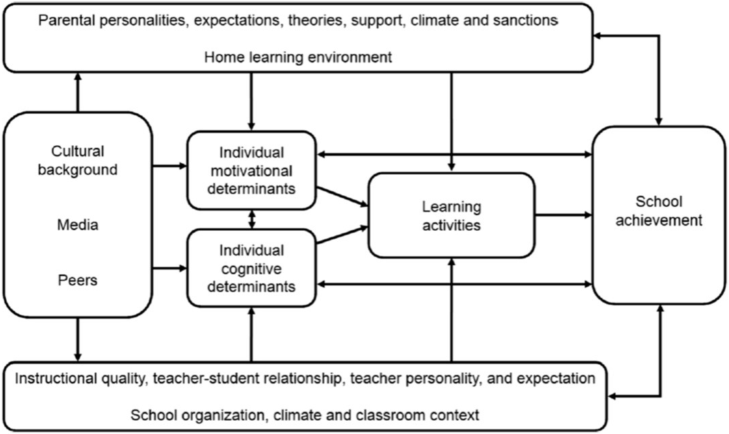
Subsection 4

Teaching

Teaching CE – ME vs core mathematics

- Different structures:
  - Different students
  - Different teachers
- Different teaching:
  - Different features in teaching CE
  - CE includes basics calculus (secondary school level).
- Different learning outcomes

Schrader & Helmke (2015), p. 49 •



Section 3

The framework of calculus in economics

3 The framework of calculus in economics

3.1. Need for CE: stocks and flows

3.2. Need for CE: further reasons

3.3. Topics of CE

Subsection 1

Need for CE: stocks and flows

Static and dynamic economics

- Static economics:
  - One point of time or one period
  - Time does not matter.
- Dynamic economics:
  - At least two points of time or at least two periods
  - Time does matter.

Discrete and continuous time

- Discrete time:
  - Sequences
  - y_t
  - $t \in \mathbb{N}$, $t \in \mathbb{N}_0$
  - Difference equations
- Continuous time:
  - Functions $y(t)$
  - $y(t)$
  - $t \in \mathbb{R}$, $t \in \mathbb{R}_0^+$
  - Differential equations
- Advantages / disadvantages:
  - Models with discrete time are easier to understand.
  - Models with continuous time are easier to handle from a mathematical point of view.

Stock and flows I (*wikipedia* (2023))

Loriot monument "Herren im Bad" (translated: gentlemen in the bath) Münsing, a sculpture based on the cartoon (left Dr. Klöbner, right Müller-Lüdenscheidt)

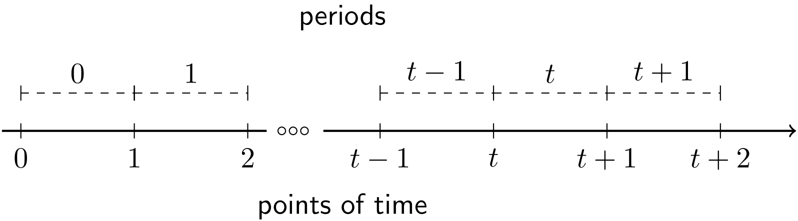
Stocks and flows II

- Example bathtub:
  - Stock at $t = 0$: $K_0 = 0$
  - Capacity: 200 liter
  - Water inflow: 50 liter per minute (on average)
  - Bathtub is filled in 4 minutes.
- Stocks and flows in economics:
  - For economics students stock and flows become supposed complicated in economics.
  - Reasons: it is economics and / or it turns to differential calculus.
- However, is everything simple with stocks and flows?
  - Why "on average"?
  - What is a flow exactly?

Stocks and flows III

- Stock:
  - At point of time t
  - Example: capital stock K_t, respectively $K(t)$
- Flow:
  - For a period t
  - Example: investment I_t, respectively $I(t)$
- Obviously, we use t for points of time and periods of time – that's a first confusion.

Discrete time – time of points t and periods t



period $0 = [0;1]$ period $1 = [1;2]$ period $t = [t;t+1]$

length of any period $= 1$ [units of time]

The relation between stocks and flows

- Discrete time:

$$K_{t+1} = K_t + I_t \qquad \Leftrightarrow \qquad K_{t+1} - K_t = I_t$$

- Continuous time:

$$\frac{dK(t)}{dt} = I(t) \qquad \Leftrightarrow \qquad dK(t) = I(t)dt$$

- Flows change stocks in time.

Accumulation

- Stock-flow models are connected with accumulation.
- Models with stocks and flows are very useful to motivate
  - differential CE and
  - integral CE.
- Example stock of resources $S(t)$ and consumption of resources $R(t)$:

$$\begin{aligned} R(t) &= \frac{dS(t)}{dt} \\ S(t) &= \int_0^t R(s)\,ds \end{aligned}$$

- However, many stumbling blocks are present.
- In particular: what are flows and how are flows measured?

Subsection 2

Need for CE: further reasons

Some motivations for differential CE

- Description and explanation of changes:
  - In time (accumulation)
  - Not in time (e.g. comparative statics)
- Properties of functions:
  - Monotonicity
  - Convexity and concavity
- Optimization:
  - Unconstrained
  - Constrained
- Approximation:
  - Linear
  - Taylor
- Cases: $n = 1$, $n = 2$, $n \in \mathbb{N}$

Some motivations for integral CE

- Description and explanations of changes:
  - In time (accumulation)
  - Not in time (e.g. comparative statics)
- Calculation of:
  - Areas and
  - Volumes
- In statistics calculation of:
  - Probabilities and
  - Expected values
- Cases: $n = 1$, $n = 2$, $n \in \mathbb{N}$

Subsection 3

Topics of CE

Differential calculus: $n = 1$

1. Difference and differential quotient
2. First, second and n-th derivative
3. Basic rules of differential calculus
4. Chain rule of differential calculus
5. Derivatives of important functions
6. Conditions for (strict) monotony, convexity and concavity
7. Conditions for (strict) minima and maxima as well as turning points
8. Differentials
9. Linear approximation
10. L'Hôpital's rules
11. Growth rates
12. Elasticities

Differential calculus: $n > 1$

1. First, second and n-th partial derivatives
2. Schwarz' theorem
3. Conditions for (strict) convexity and concavity
4. Conditions for (strict) minima and maxima as well as saddle points
5. Partial and total differentials
6. Implicit derivatives
7. Generalized chain rule
8. Partial elasticities

Integral calculus

1. Indefinite integrals
2. Definite integrals
3. Fundamental theorem of calculus
4. Basic rules
5. Integration by substitution
6. Integration by parts
7. Surface areas
8. Improper integrals
9. Multiple integrals

Outline "Mathematics for economists" (Kassel)

0. Introduction: Examples; why math?; basic math terms
1. Repetition: Geometry, arithmetics, algebra
2. Mathematical logic; modelling
3. Sets, cartesian products, (binary) relations, functions
4. Sequences, series, financial mathematics *
5. Functions (one variable) *
6. Differential calculus (one variable) **
7. Functions (two variables) *
8. Differential calculus (two variables) (incl. Lagrange) **
9. Integration (one and two variables) **
10. Difference and differential equations *
11. Linear algebra (introduction)
12. Linear algebra (extensions)

Section 4

Features of calculus in economics •

4 Features of calculus in economics •

Features (cf. *Voßkamp* (2023b))

1. Modeling
2. Simplifications
3. Heuristics
4. Applications / examples
5. Diagrams / graphical representations
6. Measurement

Subsection 1

Modeling •

Motivation

- Goal: gaining new knowledge about economic phenomena
- Staring point: a statement X on an economic phenomena
- Question: is the statement X true or false?
- Problem: economic phenomena are usually complex and complicated
- Solution: abstraction by using a (mathematical) model

The process of modeling

Model 1

An adequate model M is defined due to a set of assumption (definitions, hypotheses) $A.1$ to $A.n$:

$$M = \{A.1, \ldots, A.n\}$$

Theorem 1

A theorem is formulated related to statement of X

Proof 1

Proof of the theorem using $A.1$ to $A.n$.

Remarks

- Critical assessment of the model
- Examples: see next section and *Voßkamp* (2023a, 2023b)
- Modeling is crucial in economics.
- No fundamental difference to core mathematics (?)

Subsection 2

Simplifications •

Motivation

1. Within the framework of a module ME, calculus cannot be treated in full breadth and depth. In order to avoid that facts become too complicated and / or too complex, assumptions are often made within the framework of ME that lead to simplifications.
2. Simplifications are often reasonable and understandable for economic reasons.

Examples

Example (Simplifications)

1. *Domain D (I, I_1, I_2 intervals):*
   - $n = 1$: $D = I$ *or* $I = I_1 \cup I_2$
   - $n = 2$: $D = I_1 \times I_2$
   - $n > 2$: $D = \mathbb{R}^n$ *or* $D = \mathbb{R}_0^+ \times \ldots \times R_0^+$
2. *Only certain classes of functions are considered. In the case $n > 1$:*
   - *linear and affine functions:*
   - *polynomials*
   - *Cobb-Douglas functions*
3. *Function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.*

Remarks

- If simplifications are used, then no error is made in the strict sense.
- However, it must be checked whether the simplification is permissible in the context of the economic question / mathematical model.

Subsection 3

Heuristics •

Motivation

Heuristic definition 1 (Heuristic (cf. *Gigerenzer* (2011)))

A heuristic is a method to get valuable results quickly.

Heuristics - examples

Heuristic theorem 1 (Schwarz)

For all important functions in economics $z = f(x, y)$ holds:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem 2 (Schwarz)

Assume that a function $z = f(x, y)$ has continuous second partial derivatives on D where D is an open subset of $\mathbb{R}^2$. Then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Remarks

- It is important to show at least one case where the heuristics not work. For the example:

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases}$$

- If heuristics are used, errors are not excluded.
- A heuristic does not always lead to perfect results.
- For that reason, the use of simplifications and heuristics are different features.
- Further examples: *Voßkamp* (2023b)

Subsection 4

Applications / examples •

Motivation

- In ME, economic examples are used very intensively, whereby these are very often simple applications from microeconomics and macroeconomics.
- Two reasons play a role:
  1. Economic examples are often used to motivate methods in ME.
  2. Economic examples are also used to illustrate the advantages of mathematical methods and the mathematical in answering economic questions.
- In the next section this is illustrated by several examples.

Subsection 5

Diagrams / graphical representations •

Motivation I

- In ME, we work intensively with diagrams.
- The background is that both in economic textbooks as well as in economic research papers, diagrams are used intensively.
- Very often it is about the representation in $\mathbb{R}^2$ of the graphs of (two) functions of a variable.

Example

- The most important example is probably the price-quantity diagram, in which the inverse demand curve

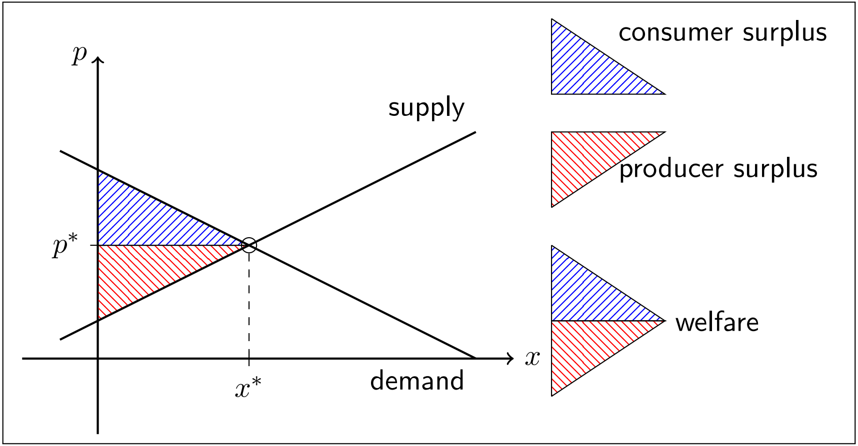
$$p = p(x; a_1, \ldots, a_m)$$

  and the inverse supply function

$$p = p(x; b_1, \ldots, b_m)$$

  are represented
- p is used for price, x for quantity. $a_1, \ldots a_m, b_1 \ldots b_m$ represent location parameters which we will disregard in the following.

Demand, supply, consumer and producer surplus, welfare



Remarks

- Market equilibrium $(x^*; p^*)$
- In this example, the slopes of the functions play an important role.
- In addition, important quantities such as consumer surplus, producer surplus and welfare can be represented by surface areas and thus by definite integrals.
- Comparative statics (using differentials):

$$\frac{dx^*}{da_i} \quad \frac{dx^*}{db_i} \quad \frac{dp^*}{da_i} \quad \frac{dp^*}{db_i} \quad \frac{dW^*}{da_i} \quad \frac{dW^*}{db_i}$$

- Details: see *Voßkamp* (2023a)

Subsection 6

Measurement •

Motivation

- For the understanding of economic matters it is (similar to physics) very often helpful to specify the units in which variables are measured.
- This is especially true in the context of marginal variables that arise by differentiation.
- As an example, the concept of marginal cost will be discussed (cf. *Feudel* (2020)).

Example I

- Assume that there is a functional relationship between the cost C and the quantity x produced.
- The cost C is measured in monetary units [$], the quantity produced x in units of quantity [pieces].
- Consequently, the marginal cost

$$C'(x) = \frac{dC}{dx}$$

is measured in monetary units per unit of quantity:

$$\frac{[\$]}{[\text{pieces}]}$$

Example II

- Explanation: consider the difference quotient or differential quotient:

$$\frac{\Delta C}{\Delta x} = \frac{C(x+\Delta x) - C(x)}{\Delta x} \qquad \frac{[\$] - [\$]}{[\text{pieces}]}$$

$$\frac{dC}{dx} = \lim_{\Delta x \to 0} \frac{C(x+\Delta x) - C(x)}{\delta x} \qquad \frac{[\$] - [\$]}{[\text{pieces}]}$$

- Marginal cost is often defined as the cost of an additional marginal unit of quantity produced.

Example III

- This formulation suggests that marginal cost is measured in monetary units [$].
- However, this is incorrect. Marginal costs ultimately indicate the average cost of an additional (marginal) unit of quantity. They are measured as in monetary units per unit of quantity: [$]/[pieces].
- This information is also important, for example, for understanding elasticities and growth rates.

Section 5

Examples for calculus in economics

5 Examples for calculus in economics

5.1. Stocks and flows: How are capital stocks and flows measured?
5.2. Growth rates: How develops a capital stock?
5.3. Elasticities: What is the impact of price on demand for a good? •
5.4. Differentials: What drives economic growth? •
5.5. Optimization: What is the impact of a monopolist's price on profit?
5.6. Improper integrals: When is an exhaustible resource exhausted?

Examples to be presented

- New examples
  - Stock and flows: How are capital stocks and flows measured?
  - Growth rates: How develops a capital stock?
- Conference paper (*Voßkamp* (2023a))
  - Elasticities: What is the impact of price on demand for a good?
  - Differentials: What drives economic growth?
  - Optimization: What is the impact of a monopolist's price on profit?
  - Impropper integrals: When is an exhaustible resource exhausted?

Further examples

- Extended conference paper (*Voßkamp* (2023b))
  - Constrained optimization: What determines demand for a good?
  - Approximation: What effect does the (expected) inflation rate have on savings decisions?
- Conference papers:
  - Frank Feudel: differential calculus in economics
  - Željka Milin Šipuš: integral calculus in economics
  - Ida Maria Landgärds: constrained optimization
  - Maria Trigueros: differential equations

Subsection 1

Stocks and flows: How are capital stocks and flows measured?

Textbook classics

- Capital stocks:

  K_t Capital stock (discrete time)
  $K(t)$ Capital stock (continuous time)

- Capital flows:

  I_t Investment (discrete time)
  $I(t)$ Investment (continuous time)

- Then, in the discrete case capital accumulation is given by:

$$K_{t+1} = K_t + I_t \quad \Leftrightarrow \quad K_{t+1} - K_t = I_t$$

- Students' conjecture on measurement:
  - Stocks K_{t+1} and K_t will be measured in [pieces].
  - Flow I_t will also be measured in [pieces].
- Is it true?

Units

- For simplification:

$$I_t = I_0 \quad \text{resp.} \quad I(t) = I_0$$

- Then:

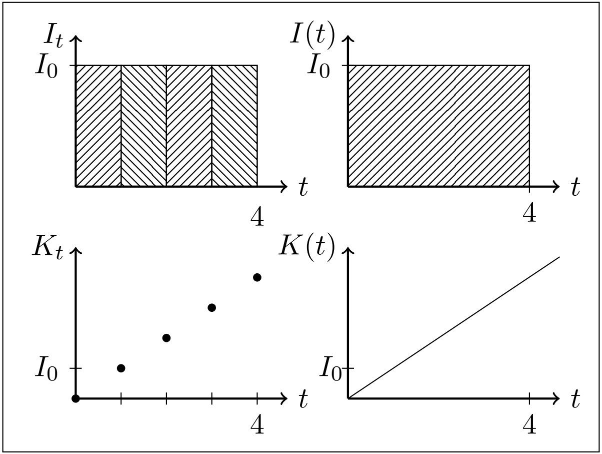
$$K_{t+1} = K_t + I_0 \quad \Leftrightarrow \quad K_{t+1} - K_t = I_0$$

- Assuming $K_0 = 0$, we have:

$$\begin{array}{cccccc} K_t & = & t & \cdot & I_0 \\ \text{[pieces]} & = & \text{[units of time]} & \cdot & \dfrac{\text{[pieces]}}{\text{[units of time]}} \end{array}$$

- Therefore: the flow I_0 has to be measured in [pieces]/[units of time]!

Graphical representation ●



Intermediate conclusion

- The conjecture is wrong!
- Why?
- There is a problem with "1".

The importance of "1"

- Obviously, I_0 indicates an average change for the period $[t; t+1]$.
- Therefore:

$$\begin{array}{ccccc} K_{t+1} - K_t & = & I_0 & \cdot & \Delta t \\ \text{[pieces]} & = & \frac{\text{[pieces]}}{\text{[units of time]}} & \cdot & \text{[units of time]} \end{array}$$

- And:

$$\begin{array}{ccccc} K_{t+1} - K_t & = & I_0 & \cdot & 1 \\ \text{[pieces]} & = & \frac{\text{[pieces]}}{\text{[units of time]}} & \cdot & \text{[units of time]} \end{array}$$

- In line with:

$$\begin{array}{ccccc} dK(t) & = & I_0 & \cdot & dt \\ \text{[pieces]} & = & \frac{\text{[pieces]}}{\text{[units of time]}} & \cdot & \text{[units of time]} \end{array}$$

Summary

- Stocks and flows are important.
- Consider discrete case and continuous case simultaneously.
- Consider units / measurement.
- Mind the 1!

Subsection 2

Growth rates: How develops a capital stock?

Textbook classics

Heuristic definition 2 (Capital growth rate)

1. *Discrete time:*

$$g_t = \frac{K_{t+1} - K_t}{K_t} \qquad \frac{\textit{[pieces]}}{\textit{[pieces]}} = \textit{[1]}$$

2. *Continuous time:*

$$g(t) = \frac{dK(t)/dt}{K(t)} \qquad \frac{[\textit{pieces}]/[\textit{units of time}]}{[\textit{pieces}]} = \frac{1}{[\textit{units of time}]}$$

[1] indicates: no dimension

Measurement

- There seems to be a contradiction.
- Once again: a period of length $\Delta t = 1$ has to be considered.

Growth rates – re-formulation

- So we need to define the growth rate g_t as follows:

$$\begin{aligned} g_t &= \text{relative change of } K \text{ in the interval } [t; t+1] \\ &= \frac{(K_{t+1} - K_t)/K_t}{\Delta t} = \frac{(K_{t+1} - K_t)/K_t}{1} \qquad \frac{1}{[\text{units of time}]} \end{aligned}$$

- And, with $\Delta t \to 0$ we get $g(t)$:

$$\begin{aligned} g(t) &= \lim_{\Delta t \to 0} \frac{(K(t+\Delta t) - K(t))/K(t)}{\Delta t} \qquad \frac{[\text{pieces}]/[\text{units of time}]}{[\text{pieces}]} \\ &= \lim_{\Delta t \to 0} \frac{\Delta K(t)/\Delta t}{K(t)} = \frac{dK(t)/dt}{K(t)} \qquad \frac{1}{[\text{units of time}]} \end{aligned}$$

- Then all fits.

Constant growth rates and approximation ●

- Continuous growth:
$$K(t) = K_0 e^{gt}$$
- Discrete growth:
$$K_t = K_0 (1+g)^t$$
- Students have many questions (e.g.):
  - Why do K_t and $K(t)$ describe similar processes?
  - What is g in K_t?
  - What is g in $K(t)$?
  - How are g in K_t and g in $K(t)$ related?
- Helpful approach: approximation

Constant growth rates and approximation •

- Discrete growth:

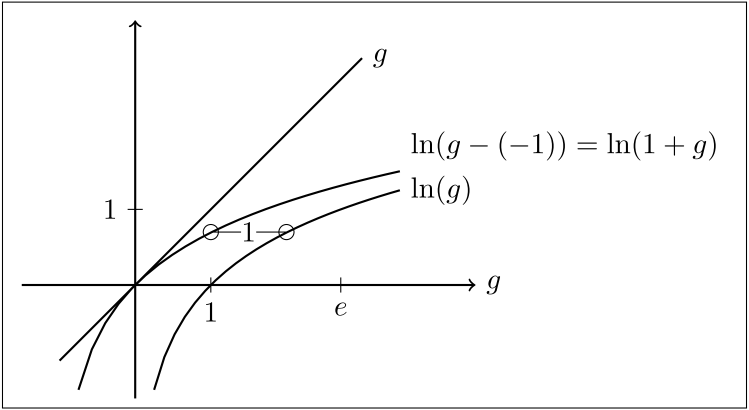
$$K_t = K_0(1+g)^t = K_0 e^{\ln(1+g)^t} = K_0 e^{\ln(1+g)t}$$

- For small g:

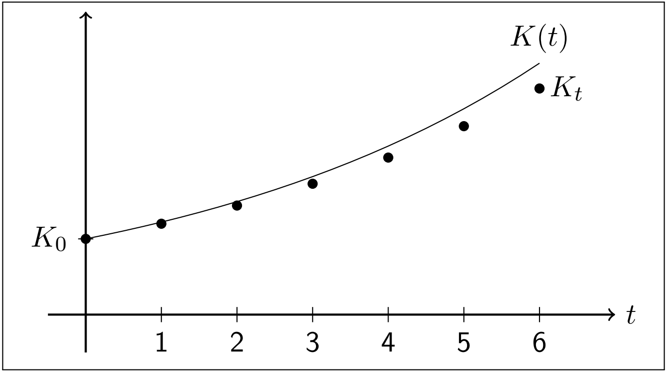
$$g \approx \ln(1+g) \qquad (= \ln(g-(-1)))$$

- Therefore:
  - Obviously, g represents growth rate in K_t (with $\Delta t = 1$)
  - g represents also a growth rate in $K(t)$
- $1+g$ is called growth factor.

Approximation $g \approx \ln(1+g)$ •



Discrete and continuous growth •



Summary

- Growth rates are important.
- Consider discrete and continuous growth rates simultaneously.
- Consider units / measurement.
- Mind the 1!
- Approximations matter.

Subsection 3

Elasticities: What is the impact of price on demand for a good? •

Motivation

- Question: What is the impact of price p on demand $x(p)$ for a good?
- Answer: Look at

$$\frac{dx(p)}{dp} = x'(p) \qquad \frac{\text{[pieces]}}{\text{[\$/pieces]}} = \frac{\text{[pieces]}^2}{\text{[\$]}}$$

- With $dx(p) = x'(p)dp$, it follows approximately with $\Delta p = 1$:

$$\Delta x \approx x'(x) \cdot 1 \qquad \text{[pieces]} = \frac{\text{[pieces]}^2}{\text{[\$]}} \cdot \frac{\text{[\$]}}{\text{[pieces]}}$$

- Comparisons for different goods (e.g. milk and cars) not meaningful (for example with $\Delta p = 1\$$)

Price elasticity – definition

Definition 2 (Price elasticity)

Assume a differentiable demand function $x = x(p)$ and $x \neq 0$ and $p \neq 0$. Then

$$\begin{aligned} \varepsilon_{x;p}(p) &= \lim_{\Delta p \to 0} \left(\frac{\Delta x}{x}\right) \Big/ \left(\frac{\Delta p}{p}\right) = \lim_{\Delta p \to 0} \left(\frac{\Delta x}{\Delta p}\right) \Big/ \left(\frac{x}{p}\right) \\ &= \left(\frac{dx}{dp}\right) \Big/ \left(\frac{x}{p}\right) \end{aligned}$$

is called the price elasticity of demand.

Example

- Example:

$$\varepsilon_{x;p}(p) \approx \frac{\Delta x/x}{\Delta p/p} = \frac{-2\ \%}{1\ \%} = -2$$

- Interpretation:
  - Assume a relative price change of 1 %.
  - Then, $\varepsilon_{x;p}(p) = -2$ implies that demand decreases approximately 2 %.
- Consequences of a macroeconomic tax shock (e.g. due to an increase of VAT) for industries are very different.

Estimates of price elasticities (cf. *Wilkinson* (2005))

product	$\varepsilon_{x;p}$
bread	-0.09
milk	$-0.18/-0.49$
tobacco products	-0.46
potatoes	-0.27
beef	$-0,65$
beer	-0.84
restaurant meals	-1.63

Elasticity – definition

Definition 3 (Elasticity)

Assume a differentiable function $y = f(x)$ *and* $y \neq 0$ *and* $x \neq 0$. *Then*

$$\varepsilon_{y;x}(x) = \left(\frac{dy}{dx}\right) \Big/ \left(\frac{y}{x}\right)$$

is called the x*-elasticity of* y.

Summary

- Elasticities are important.
- Consider discrete and continuous cases simultaneously.
- Consider units / measurement.
- Compare derivatives and elasticities.

Subsection 4

Differentials: What drives economic growth? •

Motivation

- Differentials are used very intensively in economics.
- The value of differentials can be demonstrated by the very simple model of growth accounting, which can be used to determine the main determinants of the growth rate of aggregate output.
- The starting point is a macroeconomic production function that establishes a relationship between factor inputs (labor input L and capital input K) and output Y.
- Furthermore, we assume that factor inputs change over time.

Model

Model 2 (Growth accounting)

A.1 *We assume the following production function (with $K > 0$, $L > 0$, $Y > 0$) with first partial derivatives:*

$$Y = Y(K, L)$$

A.2 *We assume (with t for time)*

$$K = K(t) \qquad L = L(t)$$

where both functions are differentiable.

Definitions

Definition 4

The growth rates of output g_Y, of capital input g_L and labor input g_L are defined as follows:

$$g_Y = \left(\frac{dY}{dt}\right) \Big/ Y \qquad g_K = \left(\frac{dK}{dt}\right) \Big/ K \qquad g_L = \left(\frac{dL}{dt}\right) \Big/ L$$

The partial elasticities of production for capital and labor, $\varepsilon_{Y;K}$ and $\varepsilon_{Y;L}$, are defined as follows:

$$\varepsilon_{Y;K} = \left(\frac{\partial Y}{\partial K}\right) \Big/ \left(\frac{Y}{K}\right) \qquad \varepsilon_{Y;L} = \left(\frac{\partial Y}{\partial L}\right) \Big/ \left(\frac{Y}{L}\right)$$

Theorem

Theorem 3 (Determinants of output growth rate)

Assuming A.1 and A.2, we have:

$$g_Y = \varepsilon_{Y,K} \cdot g_K + \varepsilon_{Y,L} \cdot g_L$$

Proof

Proof 2

A change of Y results from a change of K or L. This can be represented using the total differential:

$$dY = \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL$$

Since we are considering changes in time t, we divide by dt:

$$\frac{dY}{dt} = \frac{\partial Y}{\partial K} \frac{dK}{dt} + \frac{\partial Y}{\partial L} \frac{dL}{dt}$$

Proof (cont.)

Proof 3

Since the determinants of the growth rate of Y are to be identified, we divide by Y:

$$\frac{dY}{dt}\frac{1}{Y} = \frac{\partial Y}{\partial K}\frac{1}{Y}\frac{dK}{dt} + \frac{\partial Y}{\partial L}\frac{1}{Y}\frac{dL}{dt}$$

It follows:

$$\frac{dY}{dt}\frac{1}{Y} = \frac{\partial Y}{\partial K}\frac{K}{Y}\frac{dK}{dt}\frac{1}{K} + \frac{\partial Y}{\partial L}\frac{L}{Y}\frac{dL}{dt}\frac{1}{L}$$

Finally:

$$\frac{dY/dt}{Y} = \left(\frac{\partial Y}{\partial K}\right)\Big/\left(\frac{Y}{K}\right)\left(\frac{dK/dt}{K}\right) + \left(\frac{\partial Y}{\partial L}\right)\Big/\left(\frac{Y}{L}\right)\left(\frac{dL/dt}{L}\right) \qquad \square$$

Discussion

- The growth rate of output g_Y is taken as determined by the growth rates of factor inputs (g_K, resp. g_L) weighted by the partial elasticities of production ($\varepsilon_{Y;K}$, resp. $\varepsilon_{Y;L}$).
- Clearly, this model does not explain very much yet. Using more sophisticated models, the growth rates of factor inputs (i.e., g_K and g_L) are also explained.
- For empirical growth research, the growth accounting model plays an important role, because under certain assumptions (including profit maximization of firms) the partial production elasticities are obtained as input coefficients.

Remarks

- As in all sciences, one strives to obtain results that are as general as possible.
- This goal is achieved here to the extent that no special class of production functions (such as Cobb-Douglas production functions) needs be assumed. The production function only has to be partially differentiable once.

Summary

- Differentials important.
- Consider Δx and dx simultaneously.
- Consider units / measurement.
- Compare differentials and derivatives.

Subsection 5

Optimization: What is the impact of a monopolist's price on profit?

Economic question

- Question:
  - What is the impact if monopolist increases the price for the offered good?
- Answers:
  1. The monopolist's profit increases. O
  2. The monopolist's profit decreases. O
  3. The monopolist's profit remains constant. O
- Correct answer: It depends!

Model

Model 3 (Monopoly)

A.1 *The demand for the monopolist's good is given by the following inverse demand function (with $a, b \in \mathbb{R}^+$):*

$$p(x) = a - bx$$

A.2 *The monopolist's cost of production is given by the following cost function (with $c \in \mathbb{R}^+$):*

$$C(x) = cx$$

A.3 *The monopolist maximizes profit.*

Theorem

Theorem 4 (Pricing in a monopoly)

1. *If the current price p_0 is less than the profit-maximal price p^M, an increase of the price leads to an increase of the profit as long as $p \leq p^M$ holds.*
2. *But: if $p_0 > p^M$ holds, an increase in price will lead to a reduction in profit.*

Proof

Proof 4

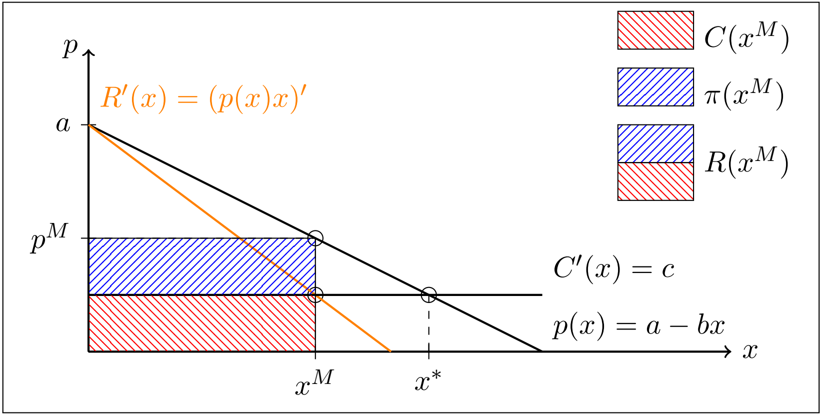
The monopolist's profit $\Pi(x)$ is given by revenues $R(x)$ minus cost $C(x)$:

$$\Pi(x) = R(x) - C(x) = p(x)x - C(x) = ax - bx^2 - cx$$

Using the known necessary and sufficient conditions for a maximum, we obtain the profit-maximum quantity x^M and profit-maximum price p^M:

$$x^M = \frac{a-c}{2b} \qquad p^M = \frac{a+c}{2}$$

Monopoly



Summary

- Modeling using differential calculus is important in economics.
- Simplifications are useful.
- Graphical representations are useful.
- Use of parameters is crucial.
- Optimization techniques are important.

Subsection 6

Improper integrals: When is an exhaustible resource exhausted?

Question

- Question:
  - We consider a resource (e.g. crude oil) whose stock at time $t = 0$ is given by $B_0 > 0$.
  - We assume that the resource is consumed at a constant rate r.
  - Let the consumption $R(t)$ at time $t = 0$ be $R(0) = R_0$.
  - Which statement is true?
- Answers:
  1. The resource will be available forever in no case. O
  2. The resource will be available forever under certain conditions. O
  3. The resource will be available forever in all cases. O
- Correct: The second statement is true.

Integral calculus II

Model 4 (Exhaustible resources)

A.1 *We consider a resource (e.g. crude oil) whose stock at time $t = 0$ is given by $B_0 > 0$.*

A.2 *We assume that the resource is consumed at a constant rate r. Let the consumption $R(t)$ at time $t = 0$ be $R(0) = R_0$.*

Integral calculus III

Theorem 5 (Exhaustible resources)

Assume A.1 and A.2. The resource will be exhausted at time T if and only if

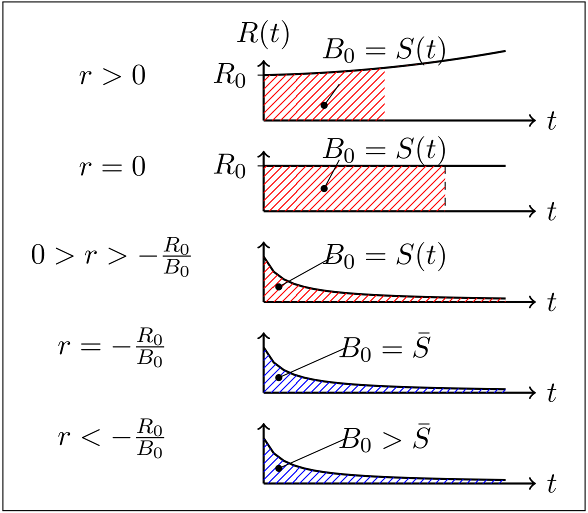
$$r > -\frac{R_0}{B_0}$$

In other cases, the resource is available everlasting.

Proof 5

See Voßkamp (2023b).

Exhaustible Resources



Integral calculus V

Example (Case study crude oil – data: statista.com)

At the begin / mid / end of 2019 (exact information is not available) there were 245 billion tons of crude oil reserves worldwide. Consumption in 2019 was 4.46 billion tons. If consumption remains constant, the reserves will be exhausted after 54.9 years. At a growth rate of

$$r = -\frac{R_0}{B_0} = -\frac{4.46}{245} = -1.8\,\%$$

the reserves are not exhausted. This result is quite important with respect to the question of crude oil availability. If consumption were to decrease slightly worldwide, crude oil would be available everlasting.

Summary

- Modeling using integral calculus is important in economics.
- Simplifications are useful.
- Graphical representations are useful.
- Use of parameters is crucial.
- Understanding of stock-flow relations is important.

Section 6

Conclusion

6 Conclusion

6.1. What is (not) different in ME?

6.2. What could be contributions of ME to others?

6.3. What could be contributions of others to ME?

6.4. Networks!

Subsection 1

What is (not) different in ME?

Rough summary

- Special students with special gaps
- Special teachers with special characteristics
- Special teaching in CE as a consequence
- Special features in ME and CE

Topics discussed

1. Models with discrete versus continuous time
2. Stock-flow models
3. Growth rates
4. Elasticities
5. Differentials
6. Approximation

Topics not discussed yet

1. Constrained optimization
2. Difference and differential equations
3. To be. cont.

Features discussed

1. Modeling
2. Simplifications
3. Heuristics
4. Applications / examples
5. Diagrams / graphical representations
6. Measurement

Further features not discussed yet

1. Use of parameters (in applications and examples)
2. Only simple proofs
3. Importance of difference and differential equations
4. To be cont.

Differences

- Features are also present in core mathematics and in other disciplines.
- Level of usage seems to be different.
- Summarizing, at undergraduate (bachelor and master) level:
  - ME (and CE) is pragmatic.
  - KISS: Keep it simple, stupid!
- At the graduate level: things are different.

Behind the horizon of undergraduate ME

- In some research areas of economics undergraduate ME is sufficient.
- However, in the others "mathematical economics" play an important role (e.g. *Arrow* (1981, 1982, 1986, 1991)).
- Mathematical economics is close to core mathematics.
- Therefore, a gap between "undergraduate ME" and "graduate ME" exists.

Subsection 2

What could be contributions of ME to others?

Contributions of ME to others

- Examples for teaching mathematics pragmatically.
- Examples for the use of calculus in (economic) applications.
- Examples for applying meaningful (economic) examples.

Subsection 3

What could be contributions of others to ME?

Contributions of others to ME

- Feedback on teaching ME:
  - Students surveys: usefulness limited
  - Feedback from colleagues: rare
- Teachers of ME are widely not able to evaluate their teaching outcomes, e.g.:
  - Level of mathematical correctness
  - Quality of didactic concepts
  - Performance of students
- Need for:
  - Exchange about mathematics
  - Exchange about didactics

Subsection 4

Networks!

Networks

- Need for interdisciplinary networks and sub-networks
- Actors of a sub-network ME
  - Teachers for ME
  - Mathematicians
  - Didacticans
  - Economists
- Few networks / initiatives in Germany (selection):
  - khdm – Centre for Higher Mathematics Education
  - LLV.HD – Project group Teaching-learning Alliances in Mathematics Containing Study Programs
  - Oberwolfach Workshop – Mathematics in Undergraduate Study Programs: Challenges for Research and for the Dialogue between Mathematics and Didactics of Mathematics (2014)
- Potentials are not fully exploited.

Networks

- The founding, formation and development of successful networks is complex and complicated.
- Example: Regional innovation networks
  - *Voßkamp* (2004), *Soete/Voßkamp* (2004), *Eickelpasch et al.* (2005)
- In general, three (amongst many) necessary conditions for successful networks:
  - Sufficient trust
  - Sufficient time
  - Sufficient openness
- The calculus conference 2023 is a hotspot for the founding and development of successful (existing) networks.

Thank you very much for your attention!

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Section 7

Reaction by Frank Feudel

Question‘s for discussion to the talk „Calculus in mathematics for economics“ by Apl. Prof. Rainer Voßkamp

Dr. Frank Feudel, Humboldt-Universität zu Berlin

There are sometimes differences between the exact mathematics from a mathematician's point of view, and the way mathematical concepts are used and interpreted in economics, e.g.,

- Interpretation of the derivative $C'(x)$ as *cost of the next (/last) unit*
- Consideration of continuous functions as functions that can be drawn without lifting the pencil

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Question 1:

To what extent should economics students become aware of such differences, and how can one try to bridge these?

Discrete and continuous time

- Discrete time:
  - Sequences
  - y_t
  - $t \in \mathbb{N}$, $t \in \mathbb{N}_0$
  - Difference equations
- Continuous time:
  - Functions $y(t)$
  - $y(t)$
  - $t \in \mathbb{R}$, $t \in \mathbb{R}_0^+$
  - Differential equations
- Advantages / disadvantages:
  - Models with discrete time are easier to understand.
  - Models with continuous time are easier to handle from a mathematical point of view.

Question 2:

To what extent does this distinction also play a role in other modelling contexts, for example contexts involving goods?

Model 2 (Monopoly)

A.1 *The demand for the monopolist's good is given by the following inverse demand function (with $a, b \in \mathbb{R}^+$):*

$$p(x) = a - bx$$

A.2 *The monopolist's cost of production is given by the following cost function (with $c \in \mathbb{R}^+$):*

$$C(x) = cx$$

Cobb-Douglas-production function:

$$f(x_1, x_2) = A \cdot x_1^a \cdot x_2^b$$

(Varian, 2014, p. 356)

Question 3:

Are the function classes chosen for contextual reasons or for didactical reasons (or a combination thereof)?

Model 2 (Monopoly)

A.1 *The demand for the monopolist's good is given by the following inverse demand function (with $a, b \in \mathbb{R}^+$):*

$$p(x) = a - bx$$

A.2 *The monopolist's cost of production is given by the following cost function (with $c \in \mathbb{R}^+$):*

$$C(x) = cx$$

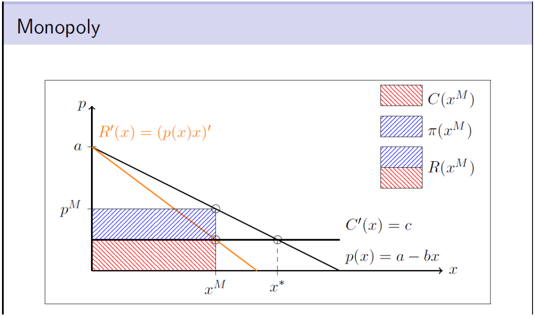
Cobb-Douglas-production function:

$$f(x_1, x_2) = A \cdot x_1^a \cdot x_2^b$$

(Varian, 2014, p. 356)

Question 4:

What is the role of such parameters in economics, and to what extent is their meaning also discussed? What might be suitable consequences for teaching calculus?



Question 5:

Why and to what extent are these geometric methods used for approaching economics problems? What is the relationship to analytic methods?

Proof 1

A change of Y results from a change of K or L. This can be represented using the total differential:

$$dY = \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL$$

Since we are considering changes in time t, we divide by dt:

$$\frac{dY}{dt} = \frac{\partial Y}{\partial K}\frac{dK}{dt} + \frac{\partial Y}{\partial L}\frac{dL}{dt}$$

Question 6:

What could be a suitable conception of differential used in economics?

Thank you very much Rainer!