Calculus in engineering

Issues and opportunities for instruction in the 2020’s
My background

• Undergraduate double major physics & mathematics
  • Rings and fields was awesome
• High school physics and calculus teacher
• Electrical Engineering PhD
  • Electromagnetic fields coursework, engineering education research
• Electrical engineering professor
  • Circuit theory and motors classes
  • Intro to applied algebra
  • American Society for Engineering Education (ASEE) mathematics division
Part 1: A history of Calculus

(according to engineers)
Calculus was invented to study the natural world ~1670 with infinitesimals
Infinitesimal calculus used to develop beam theory ~1760 and hydrodynamics ~1740.
Wierstrauss publishes modern $\varepsilon\delta$ formalism in 1862

*Some historians claim Cauchy invented them in ~1820*
Eiffel tower constructed with beam theory
1887
Present day: Engineering use of calculus continues in the Bernoulli tradition

\[ e^{-\infty} \approx 0 \]

\[ (\Delta x)^2 + \Delta x \approx \Delta x \]
Engineering sub-disciplines

- Materials
- Agricultural
- Computer Science
- Biomedical
- Computer Engineering
- Aerospace
- Nuclear
- Construction Management
- Civil
- Mechanical
- Electrical
- Chemical
Part 2: Calculus in engineering education
Physicists’ radar spurs the heavy calculus core in engineering

• Engineers as technicians -> Engineers as professionals
Calculus is foundational to nearly all engineering theory.
One failure in calculus is devastating

• "Calculus is a pre-requisite for mathematical maturity more than just the actual calculus"*

“When am I ever going to use this?”
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When a problem in a lesson applies calculus knowledge, add a dot.

\[ V(x) = \int_{x=0}^{x=L} w(x) \, dx \]
Alternate path applies calculus knowledge, add an empty dot.
8% of Statics problems use calculus
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Part 3: Limits in engineering
Only very simple limits are evaluated in engineering

- Big O notation in Computer Science
- Sinc(t) = \frac{\sin(t)}{t} function in signal processing

\[ e^{-\infty} \approx 0 \]

\[ (\Delta x)^2 + \Delta x \approx \Delta x \]

\[ |H(\omega = \infty)| = \lim_{\omega \to \infty} \frac{\omega}{\sqrt{\omega^4 + A\omega^2 + B}} \approx \frac{\omega}{\sqrt{\omega^4 + A\omega^2}} \approx \frac{\omega}{\sqrt{\omega^4}} \approx \frac{1}{\infty} \approx 0 \]
Continuity represents important physical properties.
Electric field of a line charge

For "very large" z

\[ E_z = k \lambda z \int_{-a}^{b} \frac{dx}{(z^2 + x^2)^{3/2}} = \frac{k \lambda}{z} \left[ \frac{x}{(z^2 + x^2)^{1/2}} \right]_{-a}^{b} \]

For "very small" z

\[ E_z = \frac{k \lambda}{z} \left[ \frac{b}{(z^2 + b^2)^{1/2}} + \frac{a}{(z^2 + a^2)^{1/2}} \right] \]
Electric field of a line charge

For “very large” $z \gg a, b$

$$E_z = k \lambda z \int_{-a}^{b} \frac{dx}{(z^2 + x^2)^{3/2}} = \frac{k \lambda}{z} \left[ \frac{x}{(z^2 + x^2)^{1/2}} \right]_{-a}^{b}$$

$$E_z = \frac{k \lambda}{z} \left[ \frac{b}{(z^2 + b^2)^{1/2}} + \frac{a}{(z^2 + a^2)^{1/2}} \right]$$

For “very small” $z \ll a, b$

$$E_z = \frac{2k\lambda}{z}$$

Infinite line charge result

Point charge result

$$E_z = \frac{k[\lambda(a + b)]}{z^2} \approx \frac{kQ}{r^2}$$
The continuum is an (excellent) approximation!
Engineers use impulse/delta functions

• Awkward to describe with normal calculus, but too useful to forgo
Part 4: Derivatives in engineering
Early engineering courses require only very basic derivatives.

\[
\frac{d}{dt} 5e^{-3t} \quad \text{\checkmark}
\]

\[
\frac{d}{dx} \sin(\cos(\tan(x))) \quad \text{\xmark}
\]
Engineering students can perform derivatives, but struggle to interpret them

\[
\frac{d}{dx} \ln(x) = \frac{1}{x} \quad \frac{d}{dr} \frac{\lambda}{2\pi\epsilon} \ln(r) = ?
\]
A typical homework problem:

The current flowing through an 300 mH inductor is

\[ i(t) = 2\text{[mA]} \sin(10.7 \times 10^6\text{[rad/s]}t) \]

Compute the resulting terminal voltage using

\[ v(t) = L \frac{d}{dt} i(t) \]
During what interval is the greatest voltage applied to the 308 H inductor the greatest?

\[ v(t) = L \frac{d}{dt} i(t) \]
Where does the atom settle in the Lennard Jones potential, in terms of the fixed constants?

\[ U(r) = 4\varepsilon \left[ \left( \frac{r}{\sigma} \right)^{-12} - \left( \frac{r}{\sigma} \right)^{-6} \right] \]
Functions in engineering are...

• Simple (lines, sines, exponentials, logs, etc)
• Piecewise-defined
• Have units and prefixes on quantities
• Must be interpreted graphically
Imagine mathematics is an island
Imagine mathematics is an island.
Imagine mathematics is an island

\[ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{d}{dt} Ke^{-\alpha t} \]

\[ \vec{F} \times \vec{r} \]
Imagine mathematics is an island

\[
\begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix} \frac{d}{dt} Ke^{-\alpha t}
\]

\(\forall x \exists y \in \{\ldots\}\)
Part 5: Integrals in engineering
At the surface of the water, is $P(h)$ discontinuous, pointed, or smooth?

\[
\frac{dP}{dh} = -\rho g
\]

$P_{\text{surface}} = 100 \text{ kPa}$

$\rho_{\text{water}} = 1000 \frac{\text{kg}}{\text{m}^3}$

$\rho_{\text{air}} = 1.2 \frac{\text{kg}}{\text{m}^3}$
Compute the energy to turn on the MOSFET

- A MOSFET is an electronic switch. A gate charge of 8 nC is needed to turn the switch on, which requires an input of electrical energy by the equation:

\[ E = \int_{q=0}^{q=8\text{nC}} V(q) dq. \]
Does the engineering solution “use calculus?”

\[
\frac{1}{2} \cdot 2 \cdot 8 + 6 \cdot 8 + \frac{1}{2} \cdot 4 \cdot 8
\]

\[= 72 \left( n \langle C \rangle \langle V \rangle \right) \]

\[72 \text{ nJ} \]
Which bending moment (antiderivative) is correct?
“Adding up pieces” and “accumulation from rate”

- Informal infinitesimals are everywhere, continuing as Bernoulli did
- Integrand and differential have units
- Units inform construction of integral
- 3D integrals in advanced classes

*Rob Ely, Infinitesimals-based registers for reasoning with definite integrals*
Part 6: Future outlook
Published in 1985:

• “To be effective and useful the design of mathematics courses for engineering students must involve a continuous and informed dialogue between engineering and mathematics departments to which each must contribute fully. The process of dialogue is essential since neither must be the dominant partner. The difficulties usually arise not in deciding what is to be taught but how and at what level. This is where the engineering department must have a clear understanding of what is needed and be able to communicate this effectively to the mathematicians.”

Are applications the answer?

• Easy to demand, hard to deliver
• Application tasks in textbooks are few and inauthentic*
• Outside domain knowledge of many math faculty
• Some promising work with model-eliciting-activities (MEAs)

The “paradox of application”

“Any application problem that a teacher picks will likely be outside the interest and field of almost all students, thus providing one more piece of evidence that they will never use that mathematical topic.”

“Teachers are forced to do the very hard work of finding or creating application problems that are general enough and compelling enough to interest all students.”

COREY, D. When Will I Ever Use This? An Essay for Students Who Have Ever Asked This Question in Math Class. Math Horizons, 22(2), p. 34, 2018
My experience teaching circuits-for-nonmajors contrasts this view

- Also wide, unmotivated audience
- Requires substantial help from colleagues in client disciplines
- Need to feel some are “just for them”
What future lies ahead?

• How will calculus instruction evolve as society and technology evolve?
• How can client disciplines more productively communicate their needs?
• A custom calculus for every major is impractical
• Can the “standard” curriculum be changed
  • AP test
  • Transfer credits
  • Prestige of calculus
5 Provocative questions for post-discussion
Provocative question 1: What is the square root of seventeen?
Provocative question 1: What calculus can we let the machines do?

\[ \sqrt{17} \]

\[ \begin{array}{c|ccc}
4 & 1 & 7 & 0 \\
+ & 4 & & \\
\hline
8 & 1 & 1 & 7 \\
+ & 1 & & \\
\hline
19 & 1 & 8 & \\
+ & 2 & & \\
\hline
21 & 3 & 6 & \\
+ & 2 & & \\
\hline
8 & 7 & 1 & \\
\end{array} \]

```
>> sqrt(17)
ans =
4.1231
```

“four and a bit” is an EXCELLENT, INSIGHTFUL answer
Are these so different?

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)
\]
Partial fractions expansion is like square roots

\[ y = \frac{2x^2 - x + 4}{x^3 + 4x} \]
\[ y1 = \text{int}(y) \]
\[ y2 = \text{int}(\text{ypf}) \]

\[ y = \frac{2x^2 - x + 4}{x^3 + 4x} \]
\[ \text{ypf} = \frac{x - 1}{x^2 + 4} + \frac{1}{x} \]
\[ y1 = \log(x) + \log(x - 2i) \left( \frac{1}{2} + \frac{1}{4}i \right) + \log(x + 2i) \left( \frac{1}{2} - \frac{1}{4}i \right) \]
\[ y2 = \frac{\log(x^2 + 4)}{2} - \frac{\arctan \left( \frac{x}{2} \right)}{2} + \log(x) \]

**Form of the rational function**

<table>
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<th>No.</th>
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<th>Form of the partial fraction</th>
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<td>1.</td>
<td>( \frac{px + q}{(x-a)(x-b)} )</td>
<td>( \frac{A}{x-a} + \frac{B}{x-b} )</td>
</tr>
<tr>
<td>2.</td>
<td>( \frac{px + q}{(x-a)^2} )</td>
<td>( \frac{A}{x-a} + \frac{B}{(x-a)^2} )</td>
</tr>
<tr>
<td>3.</td>
<td>( \frac{px^2 + qx + r}{(x-a)(x-b)(x-c)} )</td>
<td>( \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} )</td>
</tr>
<tr>
<td>4.</td>
<td>( \frac{px^2 + qx + r}{(x-a)(x^2 + bx + c)} )</td>
<td>( \frac{A}{x-a} + \frac{Bx + C}{x^2 + bx + c} )</td>
</tr>
<tr>
<td>5.</td>
<td>( \frac{px^2 + qx + r}{(x-a)(x^2 + bx + c)} )</td>
<td>( \frac{A}{x-a} + \frac{Bx + C}{x^2 + bx + c} )</td>
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where \( x^2 + bx + c \) cannot be factorised further.
If some computations are automated, what do we have more room for?

• Sensemaking with answers (reasonable sign, reasonable magnitude, etc)
• Making approximations prudently
• Comparing results
• Simulation methods
• Examining limiting cases (not limits)
Provocative Question 2: What would happen if we delayed limits to the second year?
What do students get from studying limits before derivatives?

- Even the example of ‘formal use of limits’ is not very formal, from a mathematical point of view (is any formalism needed?)
- How much from limits do engineers need? (is method of exhaustion enough?)
- Even, this use of limits, is only for particular courses? (i.e. signal processing)
- Do practicing engineers work with manipulation of limits? (no)
- Could we use infinitesimals instead of limits?
- Do students have the mathematical maturity to really get anything from study of limits?
- What forms are really necessary? \( \lim_{x \to \infty} \frac{1}{\sqrt{1 + x^2}} \)
Definitions (rigor) varies between communities

Definitions in mathematics
Definitions (rigor) varies between communities

Definitions in mathematics

Definitions in engineering
Provocative question 3: How could the topical content of calculus be rearranged?

• Could we front-load the content that is useful to all audiences earliest in time?
• The root test for convergence, for example
• What can be delayed entirely to electives and graduate level courses?
• What can be eliminated from standard instruction entirely?
• Could first-order linear constant coefficients be pulled earlier?
Provocative Question 4: What topics and techniques can be eliminated from full-stream calculus instruction?
• Suggestions of things that can be trimmed and activities that can be integrated. Good activity: the one useful thing you learn in calc II is Taylor series. Lots of intuitive and interpretive activity.

• Need understanding of math but not super formal.

• **evaluative skills.**

• The more maths can understand how engineers think, more we think how we can provide to students. Super quick, informal, interpretive reasoning. Need conceptual understanding to
Identifying absurd answers

Mathematical work creates answer
  Algorithmic procedures
  Symbol Manipulations

ANSWER

Orthogonal physical reasoning
To check answer
  Order of magnitude
  Scales wrong
  Sign is physically impossible
Things I’d eliminate

• Quotient rule
• Tests for convergence (except comparison test)
• Techniques of integration (except by parts)
• The only useful thing you learn in Calc II is Taylor series, even with its devastating DFW rate
Provocative Question 5: How much algebra is used on the job?
• Some courses in engineering say they put calc in for algebra fluency alone.

• Third discontinuity: do professionals do this algebra in practice? How much algebraic fluency do you need?

• “If you want to succeed in engineering, take more algebra. If you want to be admitted to a good engineering school, take calculus"
Engineers use “negligibly small” much more

First four digits from multimeter

5 and 6 are on thermometer